Dynamic Systems and Discrete Curve Shortening Flow

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Outline

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Definitions

Dynamic System

a system whose state changes over time, like the system of differential equations

$$\dot{x_1} = f_1(x_1, x_2)$$

$$\dot{x_2} = f_2(x_1, x_2)$$

Fixed Point

a point that is maped to itself by the function, or where for some $x^* = (x_1{}^*, x_2{}^*)$:

$$\dot{x_1} = f_1(\mathbf{x^*}) = 0$$

$$\dot{x_2} = f_2(\mathbf{x^*}) = 0$$

Definitions
cont.

Phase space diagram

the space where all states and solutions are represented

Phase portrait

trajectories of solutions plotted on the phase space

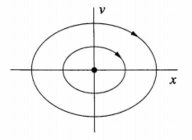

Figure: Phase portrait of harmonic oscillator

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Definitions and Examples

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- Harmonic oscillator:

$$\dot{x} = v$$
$$\dot{v} = -\frac{k}{m}x$$

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- If this is a 2 dimensional system, then often there are two eigenvalues and two eigenvectors, so the solutions are of the form $\mathbf{x} = \mathbf{c_1}\mathbf{e}^{\lambda_1 \mathbf{t}}\mathbf{v_1} + \mathbf{c_2}\mathbf{e}^{\lambda_2 \mathbf{t}}\mathbf{v_2}$.

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  - Linear systems are well behaved, so the behavior at fixed points determines the behavior elsewhere
- Real eigenvalues: both positive, both negative, or one of each in 2D case:

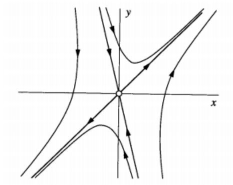

Figure: Phase portrait of a linear system

Eigenvalues and Solutions
cont.

- Complex eigenvalues lead to different behaviors about fixed points

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cont.

- Complex eigenvalues lead to different behaviors about fixed points
- Purely imaginary eigenvalues lead to "orbits", while complex eigenvalues lead to spirals

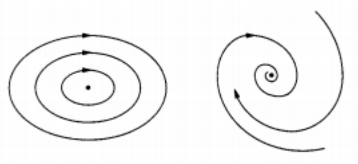

Figure: Phase portraits of linear systems with complex eigenvalues

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Definitions and Examples

Non-Linear System

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- Typically almost impossible to analytically find trajectories

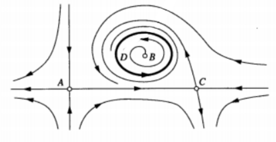

Figure: Hypothetical phase portrait of a nonlinear system

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- These approximations are analyzed like with linear systems

Example of Solving a Non-Linear System

- Consider a system describing the population growth of rabbits and sheep

$$\dot{x} = x(3 - x - 2y)$$

$$\dot{y} = y(2 - x - y)$$

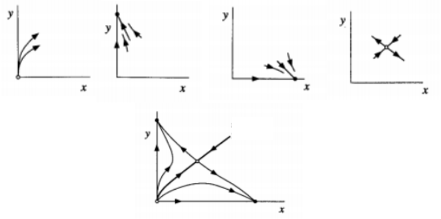

Figure: Analysis of the fixed points, and approximation of the solution

Chaos

and what makes non-linear systems difficult

- Small changes in initial conditions lead to very different results
- The Lorenz Equations:

  $\dot{x} = \sigma(y - x)$

  $\dot{y} = rx - y - xz$

  $\dot{z} = xy - bz$

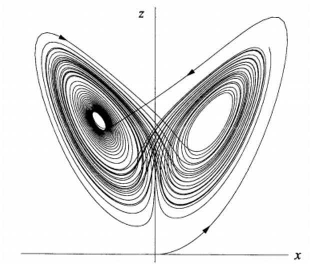

Figure: xz plane view of one trajectory

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Example

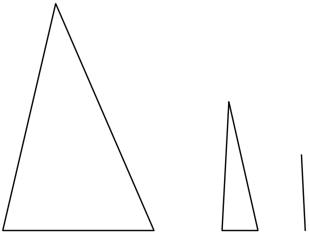

Figure: Approximate flow of a triangle under discrete curve shortening flow

Definitions

Curvature

In a discrete curve, the curvature $k(x)$ at point x is defined as $\pi - \alpha$, where α is the interior angle at x.

Normal vectors

The normal vector $\vec{n}(x)$ or $\vec{n}_x$ at point x is defined as the outward-facing unit vector in the direction of the angle bisector of the angle at x.

Differential equation

We define the motion of a point x with this differential equation:

$$\frac{dx}{dt} = -k(x)\vec{n}(x)$$

Equilateral Triangle

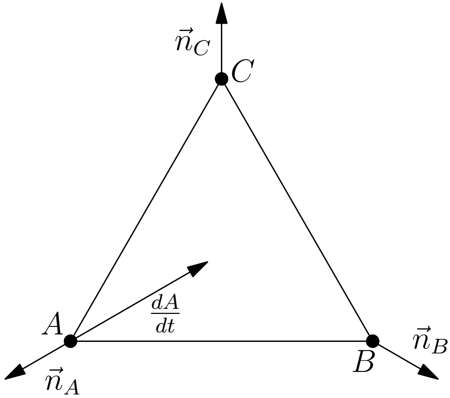

Figure: Equilateral Triangle

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Parameters

Here is how we will define the isosceles triangle:

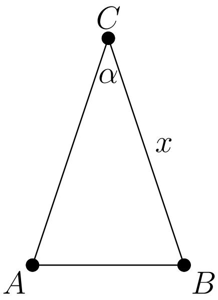
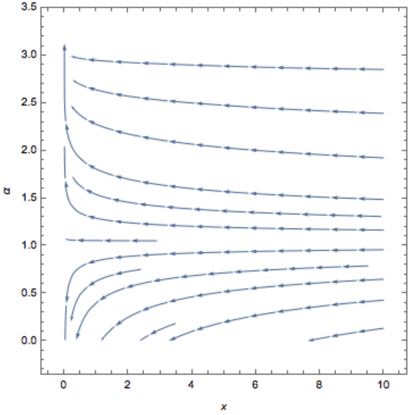


Figure: Equilateral Triangle

Finding $\frac{dx}{dt}$ and $\frac{d\alpha}{dt}$

- Finding the linear approximation of the triangle after small time ϵ, points A', B', C'
- Get differential equations using $\dfrac{df(t)}{dt} = \lim\limits_{\epsilon \to 0} \dfrac{f(t+\epsilon) - f(t)}{\epsilon}$
- $\dfrac{dx}{dt} = -\dfrac{\pi+\alpha}{2} \sin \dfrac{\pi+\alpha}{4} + \cos \dfrac{\alpha}{2}(\alpha - \pi)$
- $\dfrac{d\alpha}{dt} = \dfrac{1}{\cos\alpha} \dfrac{d \sin\alpha}{dt} = \lim\limits_{\epsilon \to 0} \dfrac{\sin\alpha' - \sin\alpha}{\epsilon}$ for $\alpha' = \angle A'B'C'$
- $\dfrac{d\alpha}{dt} = \dfrac{\frac{1}{\sqrt{2}}(\pi+\alpha)(\sin\frac{\alpha}{4} - \cos\frac{\alpha}{4}) + 2\sin\frac{\alpha}{2}(\pi - \alpha)}{x}$

Phase Plane Diagram



- horizontal axis is x, vertical axis is α

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Names and Parametrization

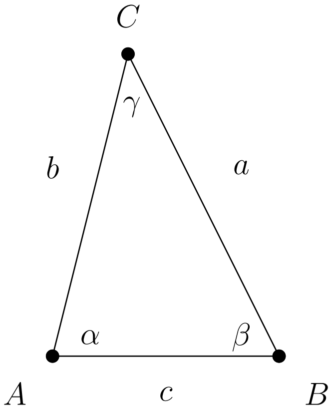

Figure: General triangle

- For parameters, we will use α, β, and c

Linear approximation

- Approximating the triangle after a small amount of time ϵ gives $\triangle A'B'C'$

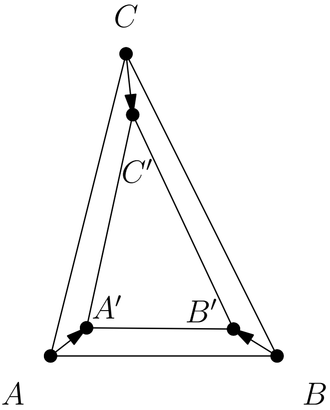

Figure: $\triangle A'B'C'$

Differential Equations

- $A' = \left(-\frac{c\cos\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon(\pi-\alpha)\cos\frac{\alpha}{2}, -\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon(\pi-\alpha)\sin\frac{\alpha}{2}\right)$
- $B' = \left(\frac{c\sin\alpha\cos\beta}{\sin(\alpha+\beta)} - \epsilon(\pi-\beta)\cos\frac{\beta}{2}, -\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)} + \epsilon(\pi-\beta)\sin\frac{\beta}{2}\right)$
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- $C' = \left(\epsilon(\alpha+\beta)\sin\frac{\alpha-\beta}{2}, -\epsilon(\alpha+\beta)\cos\frac{\alpha-\beta}{2}\right)$
- $\frac{dc}{dt} = -(\pi-\alpha)\cos\frac{\alpha}{2} - (\pi-\beta)\cos\frac{\beta}{2}$
- $\frac{d\alpha}{dt} = \frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + (\pi-\alpha)\sin\frac{\alpha}{2}\right)\sin(\alpha+\beta)}{c\sin\beta} + \frac{(\pi-\alpha)\sin\frac{\alpha}{2} - (\pi-\beta)\sin\frac{\beta}{2}}{c}$
- $\frac{d\beta}{dt} = \frac{\left(-(\alpha+\beta)\cos\frac{\alpha+\beta}{2} + (\pi-\beta)\sin\frac{\beta}{2}\right)\sin(\alpha+\beta)}{c\sin\alpha} + \frac{(\pi-\beta)\sin\frac{\beta}{2} - (\pi-\alpha)\sin\frac{\alpha}{2}}{c}$

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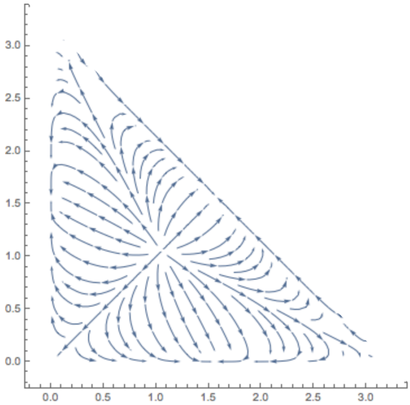

Phase plane of $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$

Summary

Dynamic Systems

- Linear systems can be solved using eigenvalues and eigenvectors
- Non-linear systems can generally not be solved directly, but their behavior can be found with linear approximations

Discrete Curve Shortening Flow

- Isosceles triangles with top angle $\geq \frac{\pi}{3}$ go to points
- All other triangles go to line segments
- Not yet proven
    - Two of the angles of any scalene triangle go to $\frac{\pi}{2}$
    - The angles go to their endpoint before c goes to 0

References

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